ON THE SCALAR CURVATURE OF CONSTANT MEAN CURVATURE HYPERSURFACES IN SPACE FORMS

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Abstract. In this paper we study the behavior of the scalar curvature S of a complete hypersurface immersed with constant mean curvature into a Riemannian space form of constant curvature, deriving a sharp estimate for the infimum of S. Our results will be an application of a weak Omori-Yau maximum principle due to Pigola, Rigoli and Setti [17].

1. Introduction

In a classical paper, Klotz and Osserman [10] characterized totally umbilical spheres and circular cylinders as the only complete surfaces immersed into the Euclidean 3-space $\mathbb{R}^3$ with constant mean curvature $H \neq 0$ and whose Gaussian curvature does not change sign. Later on, Hoffman [8] and Tribuzy [19] gave an extension of that result to the case of surfaces with constant mean curvature in the Euclidean 3-sphere $\mathbb{S}^3$ and in the hyperbolic space $\mathbb{H}^3$, respectively. Specifically, putting together the results of those authors in a single statement, one gets the following result (see also [5, Proposition 3.3]).

Theorem 1. *Let Σ be a complete surface immersed into a 3-dimensional space form $\mathbb{M}^3_c$ ($c = 0, 1, -1$) with constant mean curvature H. If its Gaussian curvature K does not change sign, then Σ is either a totally umbilical surface or $K = 0$ and*

(a) *$c = 0$ and Σ is a circular cylinder $\mathbb{R} \times \mathbb{S}^1(r) \subset \mathbb{R}^3$, with $r > 0$,*
(b) *$c = 1$ and Σ is a flat torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^1(r) \subset \mathbb{S}^3$, with $0 < r < 1$,*

Date: March 2009; revised version August 2009.

2000 *Mathematics Subject Classification.* 53C40, 53C42.

Key words and phrases. constant mean curvature, scalar curvature, Ricci curvature, maximum principle, parabolicity, stochastic completeness.

This work was partially supported by MEC projects MTM2007-64504 and Fundación Séneca project 04540/GERM/06, Spain. This research is a result of the activity developed within the framework of the Programme in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Regional Agency for Science and Technology (Regional Plan for Science and Technology 2007-2010).

S. Carolina García-Martínez was supported by a research training grant within the framework of the programme Research Training in Excellence Groups GERM by Universidad de Murcia.

(c) *$c=-1$ and Σ is a hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2})\times\mathbb{S}^1(r)\subset\mathbb{H}^3$, with $r>0$.*

As a nice application of Theorem 1, one gets the following consequence for the infimum of the Gaussian curvature of Σ.

Theorem 2. *Let Σ be a complete surface immersed into a 3-dimensional space form $\mathbb{M}^3_c$ ($c=0,1,-1$) with constant mean curvature H such that $H^2+c>0$, and let K stand for its Gaussian curvature. Then*

(i) *either $\inf_\Sigma K = H^2+c$, and Σ is a totally umbilical surface,*

(ii) *or $\inf_\Sigma K\leq 0$, with equality if and only if*

  (a) *$c=0$ and Σ is a circular cylinder $\mathbb{R}\times\mathbb{S}^1(r)\subset\mathbb{R}^3$, with $r>0$,*

  (b) *$c=1$ and Σ is a flat torus $\mathbb{S}^1(\sqrt{1-r^2})\times\mathbb{S}^1(r)\subset\mathbb{S}^3$, with $0<r<1$,*

  (c) *$c=-1$ and Σ is a hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2})\times\mathbb{S}^1(r)\subset\mathbb{H}^3$, with $r>0$.*

Actually, it follows from the Gauss equation of the surface that $K\leq H^2+c$ on Σ, with equality at the umbilical points of Σ. Therefore, $\inf_\Sigma K\leq H^2+c$ with equality if and only if Σ is totally umbilical. This proves part (i). Moreover, if $\inf_\Sigma K<H^2+c$ then it must be $\inf_\Sigma K\leq 0$ necessarily. Otherwise, one would have $K\geq\inf_\Sigma K>0$ which is not possible by Theorem 1, since the non-totally umbilical surfaces in (a), (b) and (c) are all flat. This shows that $\inf_\Sigma K\leq 0$. Finally, if equality holds, $\inf_\Sigma K=0$, then $K\geq 0$ and the result follows from Theorem 1.

Rotational surfaces show that the estimate in Theorem 2 is sharp. For instance, let us consider the Delaunay rotational surfaces in the Euclidean space. For a given constant $H\neq 0$, we may consider the family of unduloids in $\mathbb{R}^3$ with constant mean curvature H, which are given by the following parametrization

$$(s,\theta)\mapsto(x_B(s),y_B(s)\cos\theta,y_B(s)\sin\theta)$$

where $0<B<1$ and

$$\begin{aligned} x_B(s) &= \int_0^s \frac{1+B\sin(2Ht)}{\sqrt{1+B^2+2B\sin(2Ht)}}dt \\ y_B(s) &= \frac{\sqrt{1+B^2+2B\sin(2Hs)}}{2|H|}. \end{aligned}$$

The first fundamental form of these surfaces is $ds^2+y_B(s)^2d\theta^2$ and the Gaussian curvature is then

$$K_B(s,\theta)=K_B(s)=-\frac{y_B''(s)}{y_B(s)}=\frac{4H^2B(B+\sin(2Hs))(1+B\sin(2Hs))}{(1+B^2+2B\sin(2Hs))^2}.$$

Therefore, for these examples we have $\inf K_B=-4H^2B/(1-B)^2<0$, and for a given $\varepsilon>0$ there exists $0<B<1$ such that $\inf K_B=-\varepsilon<0$.

It is worth pointing out that the proof of Theorem 1 (and hence Theorem 2) strongly depends on the conformal structure of the two-dimensional surface Σ, and cannot be extended to higher dimensions. Our objective in this paper is, using an alternative approach, to extend Theorem 2 to the case of n-dimensional hypersurfaces, with $n \geq 3$ (see Theorem 3 and Corollary 4 below). As a consequence of Theorem 3, in Corollary 6 we give a generalization of Theorem 1.5 in [1] to the case of complete parabolic hypersurfaces in space forms.

Acknowledgements. The authors would like to heartily thank A. Brasil Jr., E. de A. Costa and J.M. Espinar for their valuable suggestions and useful comments during the preparation of this paper, which served to improve the paper. They also thank the referee for valuable suggestions which improved the paper.

2. Preliminaries

Let us denote by $\mathbb{M}_c^{n+1}$ the standard model of an $(n+1)$-dimensional Riemannian space form with constant curvature c, $c = 0, 1, -1$. That is, $\mathbb{M}_c^{n+1}$ denotes the Euclidean space $\mathbb{R}^{n+1}$ when $c = 0$, the Euclidean sphere

$$\mathbb{S}^{n+1} = \{x \in \mathbb{R}^{n+2} : |x|^2 = 1\} \subset \mathbb{R}^{n+2},$$

when $c = 1$, and the hyperbolic space $\mathbb{H}^{n+1}$ when $c = -1$. In this last case, it will be appropriate for us to use the Minkowskian model of the hyperbolic space. Write $\mathbb{R}_1^{n+2}$ for $\mathbb{R}^{n+2}$, with canonical coordinates $(x_0, x_1, \ldots, x_{n+1})$, endowed with the Lorentzian metric

$$\langle , \rangle_1 = -dx_0^2 + dx_1^2 + \cdots + dx_{n+1}^2. \tag{1}$$

The $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$ is the complete simply connected Riemannian manifold with sectional curvature -1, which is realized as the hyperboloid

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}_1^{n+2} : \langle x, x\rangle_1 = -1, x_0 > 0\} \subset \mathbb{R}_1^{n+2}$$

endowed with the Riemannian metric induced from $\mathbb{R}_1^{n+2}$. In order to simplify our notation, when $c = \pm 1$ we agree to denote by $\langle , \rangle$, without distinction, both the Euclidean metric on $\mathbb{R}^{n+2}$ and the Lorentzian metric (1) on $\mathbb{R}_1^{n+2}$. We also agree to denote by $\langle , \rangle$ the corresponding Riemannian metric induced on $\mathbb{M}_c^{n+1} \hookrightarrow \mathbb{R}^{n+2}$.

Let us consider $\psi : \Sigma^n \to \mathbb{M}_c^{n+1}$ an isometric immersion of an n-dimensional orientable Riemannian manifold Σ, and denote by A its second fundamental form (with respect to a globally defined normal unit vector field N) and by H its mean curvature, $H = (1/n)\mathrm{tr}(A)$. In the general n-dimensional case, instead of the curvature, it will be more appropriate to deal with the so called traceless second fundamental form of the hypersurface, which is given by $\Phi = A - HI$, where I denotes the identity operator on $\mathcal{X}(\Sigma)$. Observe that $\mathrm{tr}(\Phi) = 0$ and $|\Phi|^2 = \mathrm{tr}(\Phi^2) = |A|^2 - nH^2 \geq 0$,

with equality if and only if Σ is totally umbilical. For that reason, Φ is also called the total umbilicity tensor of Σ.

As is well known, the curvature tensor R of the hypersurface is given by the Gauss equation, which can be written in terms of Φ as

$$\begin{aligned} R(X,Y)Z &= (c+H^2)(\langle X,Z\rangle Y - \langle Y,Z\rangle X) + \langle \Phi X, Z\rangle \Phi Y - \langle \Phi Y, Z\rangle \Phi X \\ &\quad + H(\langle \Phi X, Z\rangle Y - \langle Y, Z\rangle \Phi X + \langle X, Z\rangle \Phi Y - \langle \Phi Y, Z\rangle X) \end{aligned} \tag{2}$$

for $X, Y, Z \in \mathcal{X}(\Sigma)$. In particular, the Ricci and the scalar curvatures of Σ are given, respectively, by

$$\mathrm{Ric}(X,Y) = (n-1)(c+H^2)\langle X,Y\rangle + (n-2)H\langle \Phi X, Y\rangle - \langle \Phi X, \Phi Y\rangle, \tag{3}$$

for $X, Y \in \mathcal{X}(\Sigma)$, and

$$S = n(n-1)(c+H^2) - |\Phi|^2. \tag{4}$$

2.1. **Stochastic completeness and the Omori-Yau maximum principle.** For the proof of our results in higher dimension, we will make use of a weaker version of the Omori-Yau maximum principle. Following the terminology introduced by Pigola, Rigoli and Setti in [17], the Omori-Yau maximum principle is said to hold on an n-dimensional Riemannian manifold Σ^n if, for any smooth function $u \in \mathcal{C}^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in Σ with the properties

$$(i)\quad u(p_k) > u^* - \frac{1}{k}, \quad (ii)\quad |\nabla u(p_k)| < \frac{1}{k}, \quad \text{and} \quad (iii)\quad \Delta u(p_k) < \frac{1}{k}. \tag{5}$$

In this sense, the classical result given by Omori and Yau in [15, 20] states that the Omori-Yau maximum principle holds on every complete Riemannian manifold with Ricci curvature bounded from below. More generally, as shown by Pigola, Rigoli and Setti [17, Example 1.13], a sufficiently controlled decay of the radial Ricci curvature of the form

$$\mathrm{Ric}_\Sigma(\nabla\varrho, \nabla\varrho) \geq -C^2 G(\varrho) \tag{6}$$

where ϱ is the distance function on Σ to a fixed point, C is a positive constant, and $G : [0, +\infty) \to \mathbb{R}$ is a smooth function satisfying

$$\text{(i)}\ G(0) > 0,\ \ \text{(ii)}\ G'(t) \geq 0,\ \ \text{(iii)}\ \int_0^{+\infty} 1/\sqrt{G(t)} = +\infty \text{ and}$$

$$\text{(iv)}\ \limsup_{t\to+\infty} tG(\sqrt{t})/G(t) < +\infty,$$

suffices to imply the validity of the Omori-Yau maximum principle. In particular, and following the terminology introduced by Bessa and Costa in [3], the Omori-Yau

maximum principle holds on a complete Riemannian manifold whose Ricci curvature has *strong quadratic decay* [6], that is, with

$$\mathrm{Ric}_\Sigma \geq -C^2(1+\varrho^2\log^2(\varrho+2)).$$

On the other hand, as observed also by Pigola, Rigoli and Setti in [17], the validity of Omori-Yau maximum principle on Σ^n does not depend on curvature bounds as much as one would expect. For instance, the Omori-Yau maximum principle holds on every Riemannian manifold which is properly immersed into a Riemannian space form with controlled mean curvature (see [17, Example 1.14]). In particular, it holds for every constant mean curvature hypersurface properly immersed into a Riemannian space form.

More generally, and following again the terminology introduced in [17], the *weak* Omori-Yau maximum principle is said to hold on a (not necessarily complete) n-dimensional Riemannian manifold Σ^n if, for any smooth function $u\in\mathcal{C}^2(\Sigma)$ with $u^*=\sup_\Sigma u<+\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in Σ with the properties

$$(i)\quad u(p_k)>u^*-\frac{1}{k},\quad\text{and}\quad(ii)\quad \Delta u(p_k)<\frac{1}{k}. \tag{7}$$

As proved by Pigola, Rigoli and Setti [16], the fact that the weak Omori-Yau maximum principle holds on Σ is equivalent to the stochastic completeness of the manifold (see also [17, Theorem 3.1]). In particular, the weak Omori-Yau maximum principle holds on every parabolic Riemannian manifold (see also [7, Corollary 6.4]).

3. Statement of the main results

Now we are ready to state the following extension of Theorem 2 to the case of n-dimensional hypersurfaces, with $n\geq 3$.

Theorem 3. *Let Σ^n be a stochastically complete hypersurface immersed into an $(n+1)$-dimensional space form $\mathbb{M}_c^{n+1}$ ($c=0,1,-1$ and $n\geq 3$) with constant mean curvature H such that $H^2+c>0$, and let S stand for its scalar curvature. Then*

(i) *either* $\inf_\Sigma S=n(n-1)(c+H^2)$ *and Σ is a totally umbilical hypersurface,*

(ii) *or*

$$\inf_\Sigma S\leq\frac{n(n-2)}{2(n-1)}\left(2(n-1)c+nH^2+|H|\sqrt{n^2H^2+4(n-1)c}\right).$$

Moreover, the equality holds and this infimum is attained at some point of Σ if and only if

(a) *$c=0$ and Σ is an open piece of a circular cylinder $\mathbb{R}\times\mathbb{S}^{n-1}(r)\subset\mathbb{R}^{n+1}$, with $r>0$,*

(b) *$c=1$ and Σ is an open piece of either a minimal Clifford torus $\mathbb{S}^k(\sqrt{k/n})\times\mathbb{S}^{n-k}(\sqrt{(n-k)/n})\subset\mathbb{S}^{n+1}$, with $k=1,\ldots,n-1$, or a constant mean curvature torus $\mathbb{S}^1(\sqrt{1-r^2})\times\mathbb{S}^{n-1}(r)\subset\mathbb{S}^{n+1}$, with $0<r<\sqrt{(n-1)/n}$,*

(c) *$c=-1$ and Σ is an open piece of a hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2})\times\mathbb{S}^{n-1}(r)\subset\mathbb{H}^{n+1}$, with $r>0$.*

In the particular case where Σ^n is complete (which happens, for instance, when Σ^n is properly immersed), we obtain the following consequence.

Corollary 4. *Let Σ^n be a complete hypersurface immersed into an $(n+1)$-dimensional space form $\mathbb{M}_c^{n+1}$ ($c=0,1,-1$ and $n\geq 3$) with constant mean curvature H such that $H^2+c>0$. Then*

(i) *either* $\inf_\Sigma S=n(n-1)(c+H^2)$ *and Σ is a totally umbilical hypersurface,*

(ii) *or*

$$\inf_\Sigma S\leq\frac{n(n-2)}{2(n-1)}\left(2(n-1)c+nH^2+|H|\sqrt{n^2H^2+4(n-1)c}\right).$$

Moreover, the equality holds and this infimum is attained at some point of Σ if and only if

(a) *$c=0$ and Σ is a circular cylinder $\mathbb{R}\times\mathbb{S}^{n-1}(r)\subset\mathbb{R}^{n+1}$, with $r>0$,*

(b) *$c=1$ and Σ is either a minimal Clifford torus $\mathbb{S}^k(\sqrt{k/n})\times\mathbb{S}^{n-k}(\sqrt{(n-k)/n})\subset\mathbb{S}^{n+1}$, with $k=1,\dots,n-1$, or a constant mean curvature torus $\mathbb{S}^1(\sqrt{1-r^2})\times\mathbb{S}^{n-1}(r)\subset\mathbb{S}^{n+1}$, with $0<r<\sqrt{(n-1)/n}$,*

(c) *$c=-1$ and Σ is a hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2})\times\mathbb{S}^{n-1}(r)\subset\mathbb{H}^{n+1}$, with $r>0$.*

On the other hand, it follows from (4) that $\inf_\Sigma S=n(n-1)(c+H^2)-\sup_\Sigma|\Phi|^2$. Therefore, Theorem 3 (as well as Corollary 4) can be re-written equivalently in terms of the total umbilicity tensor as follows.

Theorem 5. *Let Σ^n be a stochastically complete hypersurface immersed into an $(n+1)$-dimensional space form $\mathbb{M}_c^{n+1}$ ($c=0,1,-1$ and $n\geq 3$) with constant mean curvature H such that $H^2+c>0$, and let Φ stand for its total umbilicity tensor. Then*

(i) *either* $\sup_\Sigma|\Phi|=0$ *and Σ is a totally umbilical hypersurface,*

(ii) *or*

$$\sup_\Sigma|\Phi|\geq\alpha_H=\frac{\sqrt{n}}{2\sqrt{n-1}}\left(\sqrt{n^2H^2+4(n-1)c}-(n-2)|H|\right)>0.$$

Moreover, the equality holds and this supremum is attained at some point of Σ if and only if

(a) *$c=0$ and Σ is an open piece of a circular cylinder $\mathbb{R}\times\mathbb{S}^{n-1}(r)\subset\mathbb{R}^{n+1}$, with $r>0$,*

(b) *$c=1$ and Σ is an open piece of either a minimal Clifford torus $\mathbb{S}^k(\sqrt{k/n})\times\mathbb{S}^{n-k}(\sqrt{(n-k)/n})\subset\mathbb{S}^{n+1}$, with $k=1,\dots,n-1$, or a constant mean curvature torus $\mathbb{S}^1(\sqrt{1-r^2})\times\mathbb{S}^{n-1}(r)\subset\mathbb{S}^{n+1}$, with $0<r<\sqrt{(n-1)/n}$,*

(c) *$c=-1$ and Σ is an open piece of a hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2})\times\mathbb{S}^{n-1}(r)\subset\mathbb{H}^{n+1}$, with $r>0$.*

In particular, we get the following consequence, which gives a generalization of Theorem 1.5 in [1] to complete parabolic hypersurfaces in space forms.

Corollary 6. *Let Σ^n be a complete parabolic hypersurface immersed into an $(n+1)$-dimensional space form $\mathbb{M}_c^{n+1}$ ($c=0,1,-1$ and $n\geq 3$) with constant mean curvature H such that $H^2+c>0$, and let Φ stand for its total umbilicity tensor. Then*

(i) *either $\sup_\Sigma|\Phi|=0$ and Σ is a totally umbilical hypersurface,*
(ii) *or*

$$\sup_\Sigma|\Phi|\geq\alpha_H=\frac{\sqrt{n}}{2\sqrt{n-1}}\left(\sqrt{n^2H^2+4(n-1)c}-(n-2)|H|\right)>0$$

with equality if and only if

(a) *$c=0$ and Σ is a circular cylinder $\mathbb{R}\times\mathbb{S}^{n-1}(r)$, with $r>0$,*
(b) *$c=1$ and Σ is either a minimal Clifford torus $\mathbb{S}^k(\sqrt{k/n})\times\mathbb{S}^{n-k}(\sqrt{(n-k)/n})$, with $k=1,\dots,n-1$, or a constant mean curvature torus $\mathbb{S}^1(\sqrt{1-r^2})\times\mathbb{S}^{n-1}(r)$, with $0<r<\sqrt{(n-1)/n}$,*
(c) *$c=-1$ and Σ is a hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2})\times\mathbb{S}^{n-1}(r)$, with $r>0$.*

4. Proof of the main results

The proof of our results is based on a Simons type formula for the Laplacian of the function $|\Phi|^2$, which has already been used by several authors. For the sake of completeness, we include here its derivation, following Nomizu and Smyth [13]. A standard tensor computation implies that

$$\frac{1}{2}\Delta|\Phi|^2=\frac{1}{2}\Delta\langle\Phi,\Phi\rangle=|\nabla\Phi|^2+\langle\Phi,\Delta\Phi\rangle. \tag{8}$$

Here $\nabla\Phi:\mathcal{X}(\Sigma)\times\mathcal{X}(\Sigma)\rightarrow\mathcal{X}(\Sigma)$ denotes the covariant differential of Φ,

$$\nabla\Phi(X,Y)=(\nabla_Y\Phi)X=\nabla_Y(\Phi X)-\Phi(\nabla_YX),\quad X,Y\in\mathcal{X}(\Sigma),$$

and $\Delta\Phi:\mathcal{X}(\Sigma)\rightarrow\mathcal{X}(\Sigma)$ is the rough Laplacian,

$$\Delta\Phi(X)=\mathrm{tr}(\nabla^2\Phi(X,\cdot,\cdot))=\sum_{i=1}^n\nabla^2\Phi(X,E_i,E_i),$$

where $\{E_1,\dots,E_n\}$ is a local orthonormal frame on Σ. Observe that, in our notation, $\nabla^2\Phi(X,Y,Z)=(\nabla_Z\nabla\Phi)(X,Y)$. Let us assume that the mean curvature H is constant. In that case, $\nabla\Phi=\nabla A$, which is symmetric by the Codazzi equation of the hypersurface and, hence, $\nabla^2\Phi$ is also symmetric in its two first variables,

$$\nabla^2\Phi(X,Y,Z)=\nabla^2\Phi(Y,X,Z),\quad X,Y,Z\in\mathcal{X}(\Sigma).$$

With respect to the symmetries of $\nabla^2\Phi$ in the other variables, it is not difficult to see that

$$\nabla^2\Phi(X,Y,Z) = \nabla^2\Phi(X,Z,Y) - R(Z,Y)\Phi X + \Phi(R(Z,Y)X).$$

Thus, using the Gauss equation (2) it follows from here that

$$\begin{aligned} \text{(9)} \quad \Delta\Phi(X) &= \sum_{i=1}^{n}\big(\nabla^2\Phi(E_i,E_i,X) - R(E_i,X)\Phi E_i + \Phi(R(E_i,X)E_i)\big) \\ &= \mathrm{tr}(\nabla_X(\nabla\Phi)) - H|\Phi|^2X + (n(c+H^2) - |\Phi|^2)\Phi X + nH\Phi^2X \\ &= -H|\Phi|^2X + (n(c+H^2) - |\Phi|^2)\Phi X + nH\Phi^2X, \end{aligned}$$

where we have used the facts that trace commutes with ∇_X and that $\mathrm{tr}(\nabla\Phi) = 0$. Therefore, by (8) we conclude that

$$\text{(10)} \qquad \frac{1}{2}\Delta|\Phi|^2 = |\nabla\Phi|^2 + nH\mathrm{tr}(\Phi^3) - |\Phi|^2(|\Phi|^2 - n(c+H^2)).$$

We will also need the following auxiliary result, known as Okumura lemma, which can be found in [14] and [1, Lemma 2.6].

Lemma 7. *Let $a_1,\dots,a_n$ be real numbers such that $\sum_{i=1}^n a_i = 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}(\sum_{i=1}^n a_i^2)^{3/2} \le \sum_{i=1}^n a_i^3 \le \frac{n-2}{\sqrt{n(n-1)}}(\sum_{i=1}^n a_i^2)^{3/2}.$$

Moreover, equality holds in the right-hand (respectively, left-hand) side if and only if $(n-1)$ of the a_i's are nonpositive (respectively, nonnegative) and equal.

4.1. **Proof of Theorem 5 (or, equivalently, Theorem 3).** Since $\mathrm{tr}(\Phi) = 0$, we may use Lemma 7 to estimate $\mathrm{tr}(\Phi^3)$ as follows

$$|\mathrm{tr}(\Phi^3)| \le \frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3,$$

and then

$$nH\mathrm{tr}(\Phi^3) \ge -n|H||\mathrm{tr}(\Phi^3)| \ge -\frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3.$$

Using this in (10), we find

$$\begin{aligned} \text{(11)} \qquad \frac{1}{2}\Delta|\Phi|^2 &\ge |\nabla\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3 - |\Phi|^2(|\Phi|^2 - n(c+H^2)) \\ &\ge -|\Phi|^2P_H(|\Phi|), \end{aligned}$$

where

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|x - n(c+H^2).$$

Observe that, since $H^2 + c > 0$, the polynomial $P_H(x)$ has a unique positive root given by

$$\alpha_H = \frac{\sqrt{n}}{2\sqrt{n-1}}\left(\sqrt{n^2H^2 + 4(n-1)c} - (n-2)|H|\right).$$

If $\sup_\Sigma |\Phi| = +\infty$, then (ii) holds trivially and there is nothing to prove. If $\sup_\Sigma |\Phi| < +\infty$, then by applying (7) to the function $|\Phi|^2$ we know that there exists $\{p_k\}_{k\in\mathbb{N}}$ in Σ such that

$$\lim_{k\to\infty} |\Phi|(p_k) = \sup_\Sigma |\Phi|, \quad \text{and} \quad \Delta|\Phi|^2(p_k) < 1/k,$$

which jointly with (11) implies

$$1/k > \Delta|\Phi|^2(p_k) \geq -2|\Phi|^2(p_k)P_H(|\Phi|(p_k)).$$

Taking limits here, we get $0 \geq -2(\sup_\Sigma |\Phi|)^2 P_H(\sup_\Sigma |\Phi|)$, that is

$$(\sup_\Sigma |\Phi|)^2 P_H(\sup_\Sigma |\Phi|) \geq 0. \tag{12}$$

It follows from here that either $\sup_\Sigma |\Phi| = 0$, which means that $|\Phi| = \text{constant} = 0$ and the hypersurface is totally umbilical, or $\sup_\Sigma |\Phi| > 0$ and then $P_H(\sup_\Sigma |\Phi|) \geq 0$. In the latter, it must be $\sup_\Sigma |\Phi| \geq \alpha_H$, which gives the inequality in (ii). Moreover, assume that equality holds, $\sup_\Sigma |\Phi| = \alpha_H$. In that case, $P_H(|\Phi|) \leq 0$ on Σ, which jointly with (11) implies that $|\Phi|^2$ is a subharmonic function on Σ. Therefore, if there exists a point $p_0 \in \Sigma$ at which this supremum is attained, then $|\Phi|^2$ is a subharmonic function on Σ which attains its supremum at some point of Σ and, by the maximum principle, it must be constant, $|\Phi| = \text{constant} = \alpha_H$. Thus, (11) becomes trivially an equality,

$$\frac{1}{2}\Delta|\Phi|^2 = 0 = -|\Phi|^2 P_H(|\Phi|).$$

From here we obtain that $\nabla\Phi = \nabla A = 0$, that is, the second fundamental form of the hypersurface is parallel. If $H = 0$ (which can occur only when $c = 1$) then by a classical local rigidity result by Lawson [11, Proposition 1] we know that Σ^n is an open piece of a minimal Clifford torus of the form $\mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n}) \subset \mathbb{S}^{n+1}$, with $k = 1, \ldots, n-1$, which trivially satisfies $|\Phi| = \text{constant} = \alpha_0 = \sqrt{n}$. If $H \neq 0$ then from the equality in (11) we also obtain the equality in Okumura lemma (Lemma 7), which implies that the hypersurface has exactly two constant principal curvatures, with multiplicities $(n-1)$ and 1. Then, by the classical results on isoparametric hypersurfaces of Riemannian space forms [12, 18, 4] we conclude that Σ must be an open piece of one of the three following standard product embeddings:

(a) $\mathbb{R}^{n-1} \times \mathbb{S}^1(r) \subset \mathbb{R}^{n+1}$ or $\mathbb{R} \times \mathbb{S}^{n-1}(r) \subset \mathbb{R}^{n+1}$ with $r > 0$, if $c = 0$;
(b) $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{S}^{n+1}$, with $0 < r < 1$, if $c = 1$; and

(c) $\mathbb{H}^{n-1}(-\sqrt{1+r^2}) \times \mathbb{S}^1(r) \subset \mathbb{H}^{n+1}$, with $0 < r < 1/\sqrt{n(n-2)}$ (recall that $H^2 > -c = 1$), or $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{H}^{n+1}$, with $r > 0$, if $c = -1$.

Obviously, in all the examples above $|\Phi| = \text{constant} = \sup_\Sigma |\Phi|$. A detailed analysis of the value of the constant $|\Phi|$ for these examples shows that when $c = 0$ $|\Phi| = \sqrt{n(n-1)}|H| > \alpha_H$ for the standard products $\mathbb{R}^{n-1} \times \mathbb{S}^1(r)$, whereas $|\Phi| = \sqrt{n}|H|/\sqrt{n-1} = \alpha_H$ for the standard products $\mathbb{R} \times \mathbb{S}^{n-1}(r)$, with $r > 0$. On the other hand, when $c = 1$ we can see that

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}}\left(\sqrt{n^2H^2 + 4(n-1)} + (n-2)|H|\right) > \alpha_H$$

for the standard products $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$ if $r > \sqrt{(n-1)/n}$, whereas

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}}\left(\sqrt{n^2H^2 + 4(n-1)} - (n-2)|H|\right) = \alpha_H$$

if $0 < r < \sqrt{(n-1)/n}$. Finally, when $c = -1$ we have that

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}}\left(\sqrt{n^2H^2 - 4(n-1)} + (n-2)|H|\right) > \alpha_H$$

for the standard products $\mathbb{H}^{n-1}(-\sqrt{1+r^2}) \times \mathbb{S}^1(r)$, with $0 < r < 1/\sqrt{n(n-2)}$, whereas

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}}\left(\sqrt{n^2H^2 - 4(n-1)} - (n-2)|H|\right) = \alpha_H$$

for the standard products $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r)$, with $r > 0$. For the details, see Appendix. This finishes the proof of Theorem 5.

4.2. **Proof of Corollary 4.** As in the previous proof, instead of proving Corollary 4 we will prove its equivalent statement in terms of the total umbilicity tensor. That is, we will show that, under the assumptions of Corollary 4, it holds that

(i) either $\sup_\Sigma |\Phi| = 0$ and Σ is a totally umbilical hypersurface,

(ii) or
$$\sup_\Sigma |\Phi| \geq \alpha_H = \frac{\sqrt{n}}{2\sqrt{n-1}}\left(\sqrt{n^2H^2 + 4(n-1)c} - (n-2)|H|\right) > 0,$$
and the equality holds and this supremum is attained at some point of Σ if and only if one has (a), (b) or (c).

Obviously, if $\sup_\Sigma |\Phi| = +\infty$, then (ii) holds trivially and there is nothing to prove. If $\sup_\Sigma |\Phi| < +\infty$, then we can estimate

$$H\langle \Phi X, X\rangle \geq -|H||\langle \Phi X, X\rangle| \geq -|H||\Phi||X|^2| \geq -|H|\sup_\Sigma |\Phi||X|^2,$$

and

$$\langle \Phi X, \Phi X\rangle \leq |\Phi|^2|X|^2 \leq (\sup_\Sigma |\Phi|)^2|X|^2,$$

for $X \in \mathcal{X}(\Sigma)$. Then, by (3) we obtain for every $X \in \mathcal{X}(\Sigma)$,

$$\begin{aligned}\mathrm{Ric}(X,X) &= (n-1)(c+H^2)|X|^2 + (n-2)H\langle \Phi X, X\rangle - \langle \Phi X, \Phi X\rangle \\ &\geq \left((n-1)(c+H^2) - (n-2)|H|\sup_\Sigma|\Phi| - (\sup_\Sigma|\Phi|)^2\right)|X|^2.\end{aligned}$$

Therefore, if $\sup_\Sigma |\Phi| < +\infty$ then the Ricci curvature of Σ is bounded from below by the constant

$$C = (n-1)(c+H^2) - (n-2)|H|\sup_\Sigma|\Phi| - (\sup_\Sigma|\Phi|)^2.$$

Since Σ is complete, the classical Omori-Yau maximum principle holds on Σ and the result follows directly from Theorem 5 (or, equivalently, Theorem 3).

Remark 8. The proof of Corollary 4 has been inspired by the estimates of the Ricci curvature for submanifolds into a Riemannian space form given by Asperti and Costa in [2]. We refer the reader to that paper for other general Ricci estimates.

4.3. **Proof of Corollary 6.** For the proof of Corollary 6, first recall that the weak Omori-Yau maximum principle holds on every parabolic Riemannian manifold. Then, by the first part of Theorem 5 we obtain that either $\sup_\Sigma |\Phi| = 0$ and Σ is a totally umbilical hypersurface, or $\sup_\Sigma |\Phi| \geq \alpha_H$. Moreover, if equality holds, $\sup_\Sigma |\Phi| = \alpha_H$, then as in the proof above we have $P_H(|\Phi|) \leq 0$ and $|\Phi|^2$ is a subharmonic function on Σ which is bounded from above. Since Σ is parabolic, it must be constant, $|\Phi| = \text{constant} = \alpha_H$. The proof then finishes as in Theorem 5, by observing that the standard Riemannian products $\mathbb{R}\times\mathbb{S}^{n-1}(r)$, $\mathbb{S}^k(\sqrt{k/n})\times\mathbb{S}^{n-k}(\sqrt{(n-k)/n})$, $\mathbb{S}^1(\sqrt{1-r^2})\times\mathbb{S}^{n-1}(r)$ and $\mathbb{H}^1(-\sqrt{1+r^2})\times\mathbb{S}^{n-1}(r)$ are all parabolic. For $\mathbb{S}^k(\sqrt{k/n})\times\mathbb{S}^{n-k}(\sqrt{(n-k)/n})$ and $\mathbb{S}^1(\sqrt{1-r^2})\times\mathbb{S}^{n-1}(r)$ this is clear because they are compact. For the other cases, it follows from the fact that any standard Riemannian product $\mathbb{R}\times M$ with M compact is parabolic (see [9, Subsection 2.1]).

Appendix

In this section we will briefly compute the value of $|\Phi|$ for the standard examples which appears in Theorem 5 and Corollary 6. In the Euclidean space ($c = 0$), apart from the totally umbilical hypersurfaces, the easiest constant mean curvature hypersurfaces are the standard product embeddings of the form $\mathbb{R}^{n-k}\times\mathbb{S}^k(r)\hookrightarrow\mathbb{R}^{n+1}$, for a given radius $r > 0$ and integer $k \in \{1,\dots,n-1\}$. Its principal curvatures are given by

$$\kappa_1 = \cdots = \kappa_{n-k} = 0, \quad \kappa_{n-k+1} = \cdots = \kappa_n = \frac{1}{r},$$

and its constant mean curvature H is given by $nH = k/r$. For these examples

$$|\Phi| = \frac{\sqrt{k(n-k)}}{\sqrt{n}r} = \frac{\sqrt{n(n-k)}}{\sqrt{k}}|H| \quad \text{and} \quad \alpha_H = \frac{\sqrt{n}}{\sqrt{n-1}}|H|.$$

In particular, $|\Phi| = \alpha_H$ if and only $k = n-1$, and $|\Phi| > \alpha_H$ otherwise.

When $c = 1$, let us consider the standard immersions $\mathbb{S}^1(\sqrt{1-r^2}) \hookrightarrow \mathbb{R}^2$ and $\mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{R}^n$, for a given radius $0 < r < 1$, and take the product immersion $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$. Its principal curvatures are given by

$$\kappa_1 = \frac{r}{\sqrt{1-r^2}}, \quad \kappa_2 = \cdots = \kappa_n = -\frac{\sqrt{1-r^2}}{r},$$

and its constant mean curvature is given by

$$H = H(r) = \frac{nr^2 - (n-1)}{nr\sqrt{1-r^2}}. \tag{13}$$

In this case,

$$|\Phi| = \frac{\sqrt{n-1}}{r\sqrt{n(1-r^2)}}$$

where, by (13),

$$r^2 = \frac{2(n-1) + nH^2 \pm |H|\sqrt{n^2H^2 + 4(n-1)}}{2n(1+H^2)}$$

where we choose the sign $-$ or $+$ according to $r^2 \leq (n-1)/n$ or $r^2 > (n-1)/n$. Therefore,

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}}\left((n-2)|H| \pm \sqrt{n^2H^2 + 4(n-1)}\right),$$

where we use the same criterion for the sign. In particular, $|\Phi| = \alpha_H$ when $r^2 \leq (n-1)/n$, and $|\Phi| > \alpha_H$ when $r^2 > (n-1)/n$.

Finally, when $c = -1$ let us consider the standard immersions $\mathbb{H}^{n-k}(-\sqrt{1+r^2}) \hookrightarrow \mathbb{R}_1^{n-k+1}$ and $\mathbb{S}^k(r) \hookrightarrow \mathbb{R}^{k+1}$, for a given radius $r > 0$ and integer $k \in \{1, \ldots, n-1\}$, and take the product immersion $\mathbb{H}^{n-k}(-\sqrt{1+r^2}) \times \mathbb{S}^k(r) \hookrightarrow \mathbb{H}^{n+1} \subset \mathbb{R}_1^{n+2}$. Its principal curvatures are given by

$$\kappa_1 = \cdots = \kappa_{n-k} = \frac{r}{\sqrt{1+r^2}}, \quad \kappa_{n-k+1} = \cdots = \kappa_n = \frac{\sqrt{1+r^2}}{r},$$

and its constant mean curvature is given by

$$H = \frac{nr^2 + k}{nr\sqrt{1+r^2}}. \tag{14}$$

We are interested in the cases where $k = 1$ and $k = n-1$. Observe that when $k = 1$, $H^2 > 1$ if and only if $r < 1/\sqrt{n(n-2)}$. In that case

$$|\Phi| = \frac{\sqrt{n-1}}{r\sqrt{n(1+r^2)}} \tag{15}$$

where, by (14),

$$r^2 = \frac{2 - nH^2 + |H|\sqrt{n^2H^2 - 4(n-1)}}{2n(H^2-1)}.$$

Thus,

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}}\left((n-2)|H| + \sqrt{n^2H^2 - 4(n-1)}\right) > \alpha_H.$$

On the other hand, when $k = n-1$ we have that $H^2 > 1$ for every $r > 0$ and $|\Phi|$ is also given by (15), where now, by (14), r^2 is given by

$$r^2 = \frac{2(n-1) - nH^2 + |H|\sqrt{n^2H^2 - 4(n-1)}}{2n(H^2-1)}.$$

Therefore, in this case we have for every $r > 0$

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}}\left((n-2)|H| - \sqrt{n^2H^2 - 4(n-1)}\right) = \alpha_H.$$

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